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Bosons in a shell: Hartree solution

J Sañudo[1] **and A F Pacheco**[2]

[1] Departamento de Física, Universidad de Extremadura, 06071 Badajoz, Spain
[2] Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

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Abstract
The Hartree solution for the ground state of $N \gg 1$ bosons placed within a spherical impenetrable shell is calculated. The bosons are assumed to interact through Coulomb repulsive forces and therefore occupy a thin volume adjacent to the shell wall. The resulting density profile is compared with its parallel Thomas–Fermi solution for fermions.

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1. Introduction

The ground state of an N-boson system may constitute a bound state provided the existing forces are appropriate. Three examples of this type of system involving long-range interactions are: (a) a self-gravitating boson cluster [1–4]; (b) a boson system with repulsive interparticle forces but globally bound by a sufficiently strong long-range centre of attraction (in the case that both repulsive and attractive effects are of the Coulomb type, this would correspond to a bosonic atom [5]); and (c) if we consider that the inter-particle force is of Coulomb repulsive type, but the system is placed within an impenetrable box that acts as an external binding agent. This third scenario would correspond, for example, to a system of deuterons or alpha particles confined, at zero temperature, within a spherical cavity, with the proviso that the mean inter-particle distance resulting should be bigger than say, a few fermi, in order to avoid the short-range effects of the nuclear forces. Physical situations of light atomic nuclei confined in a small volume are frequent in nuclear-fusion physics experiments; there, however, the temperature of the samples is considerably higher. It is known that a single-particle description forms a good approximation for many quite distinct physical systems, especially if the interaction potential is long range and extends far beyond the interparticle spacing. Therefore, for the three mentioned boson systems an appropriate approach to computing the ground-state properties is the self-consistent Hartree method.

These three mentioned boson systems have their corresponding fermion counterparts in: (a) a white dwarf [6] or a neutron star [7]; (b) an ordinary atom [8]; and (c) the problem of N electrons in a shell [9, 10]. The solution for the ground state of these fermionic systems is readily provided by the semiclassical Thomas–Fermi model [8]. In (a), if the mass of the system is very large, the solution demands the use of general relativity and thus we would have to use the Oppenheimer–Volkoff equation [7].

For the first and second above-mentioned boson systems, the explicit Hartree solutions can be found in [3, 5]. In this paper we solve the third case, i.e. the ground state of $N \gg 1$ bosons located within a spherical impenetrable shell and interacting through Coulomb repulsive forces. In the following sections, from time to time we will refer to the boson density profile near the wall as the condensate or as the boundary layer distribution. The word *condensate* in this context is, of course, reminiscent of the Bose–Einstein condensation phenomenon [11]. This term is quite suitable because in our Hartree strategy, all the particles will occupy the single-particle minimum-energy wavefunction. The reason for the words *boundary-layer distribution* lies in the fact that the repulsive Coulomb force pushes the bosons against the wall forming an almost plane particle profile. In section 2, the general Hartree equation for the 1s single-particle wavefunction is presented. In section 3, the appropriate frame of reference and the most suitable non-dimensional variables are chosen. In section 4, the so-called asymptotic solution is calculated explicitly. In section 5, the two energy terms of the system are calculated. Finally, in section 6 we compare these results with their fermionic counterparts and state our conclusions.

2. Hartree equation

Suppose N identical bosons of mass m and charge $-e$ are located within an impenetrable spherical shell of radius a. N is assumed to be $\gg 1$. The Hamiltonian of this quantum-mechanical system is

$$\hat{H} = \hat{T} + \hat{V}_r \tag{1}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\vec{\nabla}_i^2 \tag{2}$$

$$\hat{V}_r = \frac{e^2}{2}\sum_{i\neq j=1}^{N}\frac{1}{|\vec{r}_i - \vec{r}_j|} \tag{3}$$

where $\hat{T}$ and $\hat{V}_r$, are the kinetic energy and potential energy operators, respectively. In these formulae and in the following equations, the radial distances from the origin, r, are assumed to be limited: $r \leqslant a$.

Let $|\psi\rangle$ be the ground-state wavefunction of $\hat{H}$, and E its energy

$$E = \langle\psi|\hat{H}|\psi\rangle. \tag{4}$$

Likewise, we will denote by T and V_r the expectation values of the corresponding operators $\hat{T}$ and $\hat{V}_r$. As we are dealing with bosons, in the Hartree approximation:

$$|\psi\rangle = |f\rangle_1|f\rangle_2\cdots|f\rangle_N \tag{5}$$

where $|f\rangle$ is the minimum-energy single-particle wavefunction, which corresponds to a 1s state. In terms of $f(r)$ which is a real function, the particle density, $n(r)$, is given by

$$n(r) = Nf^2(r) \tag{6}$$

and hence, $f(r) = \sqrt{n(r)/N}$. Note that both f and n depend only on $|\vec{r}| = r$, i.e. on the radial distance from the origin.

Using equations (5), (2) and (3), we obtain

$$T = -\frac{\hbar^2}{2m}N\int \mathrm{d}\vec{r}\; f(r)\vec{\nabla}_r^2 f(r) \tag{7}$$

$$V_r = -\frac{e}{2}N\int \mathrm{d}\vec{r}\; f^2(r)\phi(r) \tag{8}$$

with the electric potential, $\phi(r)$, given by

$$\phi(r) = -eN \int \mathrm{d}\vec{r}\,' \, \frac{f^2(r')}{|\vec{r} - \vec{r}\,'|}. \tag{9}$$

The calculation of f, from a variational perspective, can be expressed as the solution of

$$\delta\left(T[f] + V_r[f] + \lambda N \int \mathrm{d}\vec{r}\, f^2(r)\right) = 0 \tag{10}$$

with the constraint

$$\int \mathrm{d}\vec{r}\, n(\vec{r}) = N \int \mathrm{d}\vec{r}\, f^2(r) = N \tag{11}$$

where λ is a Lagrange multiplier. Using equations (7), (8) and (10), we obtain

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}_r^2 - e\phi(r)\right] f(r) = \bar{e} f(r) \tag{12}$$

with $\bar{e} = -\lambda$. Equation (12) is the eigenvalue equation of the single-particle Hamiltonian. As usual, we will express $f(r) = u(r)/r$; in terms of $u(r)$, equation (12) reads

$$-\frac{\hbar^2}{2m} u''(r) - e\phi(r)\, u(r) = \bar{e} u(r) \tag{13}$$

which is an integro-differential equation in the radial coordinate r. Here, a prime denotes differentiation with respect to r. The function $u(r)$ fulfils the boundary conditions

$$u(r = 0) = 0 \tag{14a}$$

$$u(r = a) = 0. \tag{14b}$$

From equation (13) we can obtain a pure differential equation in $u(r)$. With this aim we divide it by $u(r)$ and then apply the Laplacian operator $\vec{\nabla}_r^2$. As Poisson's equation gives $\vec{\nabla}_r^2 \phi(r) = 4\pi e^2 N u^2(r)/r^2$, we obtain

$$\left(-\frac{\hbar^2}{2m}\right)\vec{\nabla}_r^2\left(\frac{u''r)}{u(r)}\right) - 4\pi e^2 N \frac{u^2(r)}{r^2} = 0 \tag{15}$$

which leads to

$$u'''' = -\frac{2}{r}\left\{u''' - \frac{u''u'}{u}\right\} + 2\frac{u'''u'}{u} + \frac{u''^2}{u} - 2\frac{u''u'^2}{u^2} - 8\pi \frac{e^2 N m}{\hbar^2} \frac{u^3}{r^2} \tag{16}$$

which is a fourth-order differential equation. We must be aware that, having performed two additional differentiations to pass from equation (13) to (16), some information has been lost; we will discuss this point later.

Now it is convenient to define the following dimensionless variables, x and $\tilde{u}$:

$$r = bx \tag{17a}$$

$$u = \tilde{u}/(4\pi b)^{1/2} \tag{17b}$$

$$b = \frac{\hbar^2}{2me^2 N} = \frac{a_B}{2N} \tag{17c}$$

where a_B is the Bohr radius. In terms of these variables, equation (16) reads

$$\ddddot{\tilde{u}} = -\frac{2}{x}\left\{\dddot{\tilde{u}} - \frac{\ddot{\tilde{u}}\dot{\tilde{u}}}{\tilde{u}}\right\} + 2\frac{\dddot{\tilde{u}}\dot{\tilde{u}}}{\tilde{u}} + \frac{\ddot{\tilde{u}}^2}{\tilde{u}} - 2\frac{\ddot{\tilde{u}}\dot{\tilde{u}}^2}{\tilde{u}^2} - \frac{\tilde{u}^3}{x^2} \tag{18}$$

where a dot on a function represents differentiation with respect to x. The distance a expressed in terms of b is $a = x_a b$. As in this problem we assume $N \gg 1$, b is a very small distance when compared with a_B, and hence x_a is a non-dimensional large number. The fact that $x_a \gg 1$, will be exploited in the next sections. Previously defined magnitudes in their non-dimensional form are expressed by

$$\int_0^{x_a} \mathrm{d}x\, \tilde{u}^2(x) = 1 \tag{19}$$

$$n(r) = \frac{N}{4\pi b^3}\tilde{n}(x) \tag{20a}$$

$$\tilde{n}(x) = \tilde{u}^2(x)/x^2 \tag{20b}$$

$$\frac{\ddot{\tilde{u}}(x)}{\tilde{u}(x)} + \tilde{\phi}(x) = -\tilde{\varepsilon} \tag{21}$$

$$\tilde{\varepsilon} = \frac{2m}{\hbar^2}b^2\bar{e} \tag{22}$$

$$\phi(r) = \frac{eN}{b}\tilde{\phi}(x) \tag{23}$$

$$\tilde{\phi}(x) = -\frac{1}{4\pi}\int \mathrm{d}\vec{x}'\, \frac{\tilde{u}^2(x')}{x'^2|\vec{x} - \vec{x}'|}. \tag{24}$$

The two boundary conditions, equation (14), expressed in the new variables, adopt the form

$$\tilde{u}(x = 0) = 0 \tag{25a}$$

$$\tilde{u}(x = x_a) = 0. \tag{25b}$$

Due to equation (25*b*), the solution of equation (18) is not universal, in the sense that it depends on N and a through the number x_a.

3. Change of the system of reference and asymptotic regime

Due to the electric repulsion between the particles, we expect that the boson condensate will pile in a thin volume near the wall. For this reason it is natural to move the system of reference, so far at $r = 0$, to $r = a$. Thus, in dimensionless notation, we define the radial distance y as

$$y = x_a - x. \tag{26}$$

In terms of y, equation (18) adopts the form

$$\begin{aligned}\ddddot{\tilde{u}}(y) = {}& \frac{2}{x_a(1 - y/x_a)}\left\{\dddot{\tilde{u}}(y) - \frac{\ddot{\tilde{u}}(y)\dot{\tilde{u}}(y)}{\tilde{u}(y)}\right\} \\ & +2\frac{\dddot{\tilde{u}}(y)\dot{\tilde{u}}(y)}{\tilde{u}(y)} + \frac{\ddot{\tilde{u}}^2(y)}{\tilde{u}(y)} - 2\frac{\ddot{\tilde{u}}(y)\dot{\tilde{u}}^2(y)}{\tilde{u}^2(y)} - \frac{\tilde{u}^3(y)}{x_a^2(1 - y/x_a)^2}\end{aligned} \tag{27}$$

and equations (19)–(21) and equations (23), (24) become

$$\int_0^{x_a} \mathrm{d}y\, \tilde{u}^2(y) = 1 \tag{28}$$

$$n(r) = \frac{N}{4\pi b^3}\tilde{n}(y) \tag{29}$$

$$\tilde{n}(y) = \frac{\tilde{u}^2(y)}{x_a^2(1 - y/x_a)^2} \tag{30}$$

$$\frac{\ddot{\tilde{u}}(y)}{\tilde{u}(y)} + \tilde{\phi}(y) = -\tilde{\varepsilon} \tag{31}$$

$$\phi(r) = \frac{eN}{b}\tilde{\phi}(y) \tag{32}$$

$$\tilde{\phi}(y) = -\frac{1}{4\pi}\int \mathrm{d}\vec{x}'\, \frac{\tilde{u}^2(x')}{x'^2|\vec{x}_a - \vec{y} - \vec{x}'|} \tag{33}$$

and the boundary conditions, equation (25), are now:

$$\tilde{u}(y = x_a) = 0 \tag{34a}$$

$$\tilde{u}(y = 0) = 0. \tag{34b}$$

Now, we define 'the asymptotic regime', as the limit of $x_a \to \infty$. The exploitation of this limit leads to a simplification of the previous equations and, more importantly for us, to a universal solution. With this aim, let us perform the scale change

$$\tilde{u}(y) = \frac{\hat{u}(z)}{x_a^{1/3}} \tag{35}$$

$$y = x_a^{2/3} z. \tag{36}$$

From now on, we will use the z variable. This is the distance to the wall expressed in d units,

$$zd = a - r \tag{37a}$$

$$d = bx_a^{2/3} = a^{2/3}b^{1/3} = 2^{-1/3}a^{2/3}a_B^{1/3}N^{-1/3}. \tag{37b}$$

Using these new variables, equation (27) converts into

$$\ddddot{\hat{u}}(z) = \frac{2}{x_a^{1/3}(1 - z/x_a^{1/3})}\left\{\dddot{\hat{u}}(z) - \frac{\ddot{\hat{u}}(z)\dot{\hat{u}}(z)}{\hat{u}(z)}\right\}$$
$$+2\frac{\dddot{\hat{u}}(z)\dot{\hat{u}}(z)}{\hat{u}(z)} + \frac{\ddot{\hat{u}}^2(z)}{\hat{u}(z)} - 2\frac{\ddot{\hat{u}}(z)\dot{\hat{u}}^2(z)}{\hat{u}^2(z)} - \frac{\hat{u}^3(z)}{(1 - z/x_a^{1/3})^2} \tag{38}$$

and similarly, we obtain

$$\int_0^{x_a^{1/3}} \mathrm{d}z\, \hat{u}^2(z) = 1 \tag{39}$$

$$\tilde{n}(y) = \frac{1}{x_a^{8/3}}\hat{n}(z) \tag{40}$$

$$\hat{n}(z) = \hat{u}^2(z)/(1 - z/x_a^{1/3})^2 \tag{41}$$

$$\frac{\ddot{\hat{u}}(z)}{x_a^{4/3}\hat{u}(z)} + \tilde{\phi}(z) = -\tilde{\varepsilon} \tag{42}$$

$$\tilde{\phi}(z) = -\frac{1}{x_a}\left\{\int_0^z \mathrm{d}z'\, \frac{\hat{u}^2(z')}{(1 - z'/x_a^{1/3})} + \frac{1}{(1 - z/x_a^{1/3})}\int_z^{x_a^{1/3}} \mathrm{d}z'\, \hat{u}^2(z')\right\} \tag{43}$$

$$\hat{u}(z = x_a^{1/3}) = 0 \tag{44a}$$

$$\hat{u}(z = 0) = 0. \tag{44b}$$

In the limit $x_a \to \infty$ we consider a development of the type

$$\hat{u}(z) = \hat{u}_0(z) + \varepsilon\hat{u}_1(z) + \varepsilon^2\hat{u}_2(z) + \cdots \tag{45a}$$

where

$$\varepsilon = 1/x_a^{1/3} \tag{45b}$$

acts as a small parameter. Inserting equation (45a) into (38), in the leading order in ε, we obtain

$$\dddot{\hat{u}}_0(z) = 2\frac{\ddot{\hat{u}}_0(z)\dot{\hat{u}}_0(z)}{\hat{u}_0(z)} + \frac{\ddot{\hat{u}}_0{}^2(z)}{\hat{u}_0(z)} - 2\frac{\ddot{\hat{u}}_0(z)\dot{\hat{u}}_0{}^2(z)}{\hat{u}_0^2(z)} - \hat{u}_0^3(z). \tag{46}$$

The asymptotic function $\hat{u}_0(z)$ fulfils the two conditions

$$\hat{u}_0(z = \infty) = 0 \tag{47a}$$

$$\hat{u}_0(z = 0) = 0 \tag{47b}$$

and verifies

$$\int_0^\infty \mathrm{d}z\, \hat{u}_0^2(z) = 1. \tag{48}$$

Note that in this leading order in ε, the limit of integration is no longer $x_a^{1/3}$ but ∞.

In this asymptotic limit, the previously defined magnitudes adopt the form

$$\hat{n}(z) = \hat{n}_0(z) + \cdots = \hat{u}_0^2(z) + \cdots \tag{49}$$

$$\tilde{\phi}(z) = -\frac{1}{x_a} + \frac{\hat{\phi}(z)}{x_a^{4/3}} \tag{50}$$

$$\hat{\phi}(z) = -\int_0^z \mathrm{d}z'\, z'\hat{u}^2(z') - z\int_z^{x_a^{1/3}} \mathrm{d}z'\, \hat{u}^2(z') + \cdots \tag{51}$$

$$\frac{\ddot{\hat{u}}(z)}{\hat{u}(z)} + \hat{\phi}(z) = -\hat{\varepsilon} \tag{52}$$

$$\tilde{\varepsilon} = \frac{1}{x_a} + \frac{\hat{\varepsilon}}{x_a^{4/3}} \tag{53}$$

$$\hat{\phi}(z) = \hat{\phi}_0(z) + \cdots \tag{54}$$

$$\hat{\phi}_0(z) = -\int_0^z \mathrm{d}z'\, z'\hat{u}_0^2(z') - z\int_z^\infty \mathrm{d}z'\, \hat{u}_0^2(z') \tag{55}$$

$$\hat{\varepsilon} = \hat{\varepsilon}_0 + \cdots \tag{56}$$

$$\frac{\ddot{\hat{u}}_0(z)}{\hat{u}_0(z)} + \hat{\phi}_0(z) = -\hat{\varepsilon}_0. \tag{57}$$

It is worth emphasizing that the double differentiation of equation (57) reproduces equation (46). These two equations and their solution $\hat{u}_0(z)$ are universal: they do not depend on x_a. We will see later that for $z \gg 1$, $\hat{u}_0(z)$ is exponentially decreasing to zero. This implies that, in our asymptotic approximation, to substitute $x_a^{1/3}$ by ∞ is almost exact.

4. Asymptotic solution

In order to find a solution of $\hat{u}_0(z)$ by means of the Runge–Kutta method, we analyse its behaviour for small z, $\hat{u}_0(z) = \sum_n a_n z^n$. To apply this method we have to provide a_0, a_1, a_2 and a_3. From equation (47*b*), $a_0 = 0$, and from equation (57), $a_2 = 0$. Hence, our task will be to find a_1 and a_3 such that $\hat{u}_0(z)$ has no node (1s) and is normalized according to equation (48). Furthermore, from equation (57) as $\hat{\phi}_0(z = 0) = 0$, we have $\hat{\varepsilon}_0 = -6a_3/a_1$, which means that taking $a_1 > 0$ implies $a_3 < 0$. In summary, we must solve equation (46), with $a_0 = 0$, $a_1 > 0$, $a_2 = 0$ and $a_3 < 0$.

It is interesting to note that in equation (46), a_4 is undetermined; this is easily checked by performing the algebraic development of $\hat{u}_0(z)$ around $z = 0$. However, using equation (57) we easily conclude that $a_4 = \frac{1}{12}a_1$. Using this relation between the coefficients, the numerical integration of equation (46) is straightforward. The 'undetermined' behaviour of equation (46) with respect to a_4 is related to the loss of information produced when the $\vec{\nabla}_r^2$ operator was applied to equation (13), to obtain a pure differential equation of fourth order.

The Runge–Kutta numerical solution, fulfilling the above-mentioned conditions, leads to

$$\begin{aligned} a_1 &= 0.7132 \\ a_3 &= -0.2387. \end{aligned} \tag{58}$$

The function $\hat{u}_0(z)$ is plotted in figure 1. Using equation (57), we obtain

$$\hat{\varepsilon}_0 = 2.008. \tag{59}$$

Let us now perform an analytical study of how $\hat{u}_0(z)$ decays for large z. First, one can easily see that a behaviour of the type $\hat{u}_0(z) = c_1/(c_2 + z)^n$, c_1 and c_2 constant, does not fulfil equation (46). However, a solution of the type

$$\hat{u}_0(z) = A\mathrm{e}^{-cz} + B\mathrm{e}^{-3cz} + \cdots \tag{60}$$

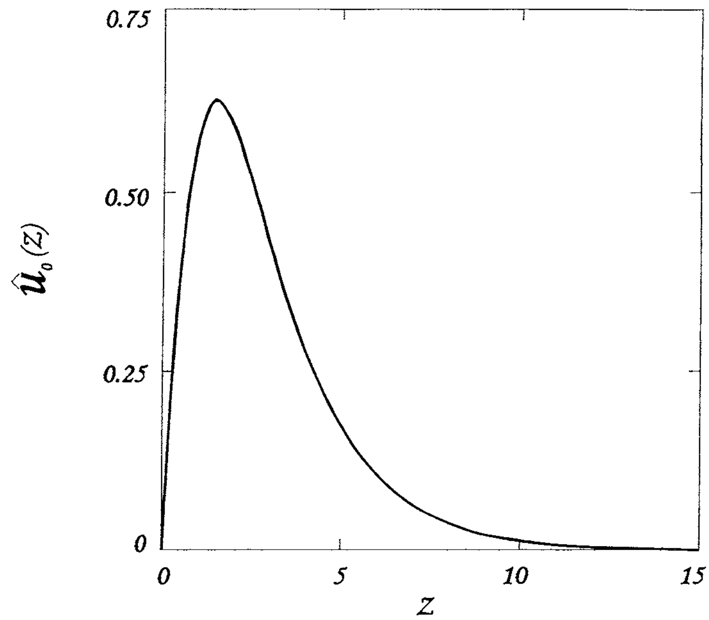

Figure 1. Plot of $\hat{u}_0(z)$ versus z.

with A, B and c constants, does fulfil equations (46) and (57) with $B = A^3/32c^4$. This result sheds light on the numerical fall off of $\hat{u}_0(z)$ for $z \to \infty$, exhibited in figure 1.

5. Energy results

The kinetic energy T is obtained from equation (7)

$$T = \frac{N^2 e^2}{b} \tilde{t} \tag{61}$$

with

$$\tilde{t} = -\int_0^{x_a} \mathrm{d}y\, \tilde{u}(y)\, \ddot{\tilde{u}}(y). \tag{62}$$

In terms of the $\hat{u}$ and z variables, we have

$$\begin{aligned} \tilde{t} &= \frac{\hat{t}}{x_a^{4/3}} = -\frac{1}{x_a^{4/3}} \int_0^{x_a^{1/3}} \mathrm{d}z\, \hat{u}(z)\, \ddot{\hat{u}}(z) \\ &\cong -\frac{1}{x_a^{4/3}} \int_0^{\infty} \mathrm{d}z\, \hat{u}_0(z)\, \ddot{\hat{u}}_0(z) = \frac{1}{x_a^{4/3}} (0.4055). \end{aligned} \tag{63}$$

The potential energy, V_r, is

$$V_r = -\frac{N^2 e^2}{2b} \tilde{v} \tag{64}$$

with

$$\begin{aligned} \tilde{v} = \int_0^{x_a} \mathrm{d}y\, \tilde{u}^2(y)\, \tilde{\phi}(y) = -\frac{1}{x_a} - \frac{\hat{v}}{x_a^{4/3}} &= -\frac{1}{x_a} + \frac{1}{x_a^{4/3}} \int_0^{x_a^{1/3}} \mathrm{d}z\, \hat{u}^2(z)\, \hat{\phi}(z) \\ &\cong -\frac{1}{x_a} + \frac{1}{x_a^{4/3}} \int_0^{\infty} \mathrm{d}z\, \hat{u}_0^2(z)\, \hat{\phi}_0(z) = -\frac{1}{x_a} - \frac{(1.6025)}{x_a^{4/3}}. \end{aligned} \tag{65}$$

In equations (63) and (65) the numbers within parentheses are obtained by numerical integration of $\hat{u}_0(z)$. Thus, T and V_r are given by

$$T = \frac{N^2 e^2}{b} \frac{(0.4055)}{x_a^{4/3}} \tag{66}$$

$$V_r = \frac{N^2 e^2}{2b} \left[\frac{1}{x_a} + \frac{(1.6025)}{x_a^{4/3}} \right] \tag{67}$$

or in terms of the small parameter ε:

$$T = \frac{N^2 e^2}{a} (0.4055) \varepsilon \tag{68}$$

$$V_r = \frac{N^2 e^2}{2a} [1 + (1.6025)\varepsilon]. \tag{69}$$

Thus, the total energy of the system, E, amounts to

$$E = T + V_r = \frac{N^2 e^2}{2a} [1 + (0.8110 + 1.6025)\varepsilon] = \frac{N^2 e^2}{2a} [1 + (2.413)\varepsilon]. \tag{70}$$

6. Comparison with fermions and conclusions

In this paper, we have analysed the structure of a boson system in a spherical impenetrable shell, interacting through repulsive Coulomb forces, at zero temperature. This analysis has been done within the Hartree approximation and the asymptotic limit, $x_a \to \infty$.

It is instructive to compare the results obtained for bosons with those resulting for fermions. We compare the two cases assuming the same number of particles, N, with identical mass and charge, m and $-e$, and the same shell radius, a. The scale length, b, for bosons and fermions, is

$$b = \frac{1}{2}\frac{a_B}{N} \tag{71a}$$

$$b_F = \kappa \frac{a_B}{N^{1/3}} \tag{71b}$$

respectively, with $\kappa = (3\pi)^{2/3}/2^{7/3}$. Note that $b/b_F \approx N^{-2/3}$. Consequently,

$$x_a = \frac{a}{b} = 2N\frac{a}{a_B} \tag{72a}$$

and

$$x_{a,F} = \frac{a}{b_F} = \frac{1}{\kappa}N^{1/3}\frac{a}{a_B}. \tag{72b}$$

And the small parameter, ε, is

$$\varepsilon = \frac{1}{x_a^{1/3}} = \left(\frac{a_B}{2a}\right)^{1/3}\frac{1}{N^{1/3}} \tag{73a}$$

and

$$\varepsilon_F = \frac{1}{x_{a,F}^{3/5}} = \left(\frac{\kappa a_B}{a}\right)^{3/5}\frac{1}{N^{1/5}}. \tag{73b}$$

With respect to the characteristic thickness of the boundary layer near the shell, we have

$$d = b x_a^{2/3} = 2^{-1/3} a^{2/3} a_B^{1/3} N^{-1/3} = a\varepsilon \tag{74a}$$

and

$$d_F = \kappa^{3/5} a^{2/5} a_B^{3/5} N^{-1/5} = a\varepsilon_F. \tag{74b}$$

With respect to the particle distribution in the boson case, the density at the wall is zero, it grows to a maximum and then falls exponentially to zero (see figure 1). In the fermion case, the particle density behaves as

$$n \propto \frac{1}{(q_0 + z)^6} \tag{75}$$

where q_0 is a constant and z is the distance to the wall expressed in d_F units.

Finally, with respect to the energy of the system up to the first-order term in ε, we have

$$V_r = \frac{N^2 e^2}{2a}[1 + 1.6025\varepsilon] \tag{76a}$$

$$V_{r,F} = \frac{N^2 e^2}{2a}[1 + 0.4859\varepsilon_F] \tag{76b}$$

and

$$T = \frac{N^2 e^2}{a}(0.4055)\varepsilon \tag{77a}$$

$$T_F = \frac{D}{\kappa^{3/5}} \frac{N^2 e^2}{a} \varepsilon_F \tag{77b}$$

with $D = \frac{1}{24}(60\pi)^{2/5}$. Hence

$$\frac{E}{E_F} = \frac{1 + 2.413\varepsilon}{1 + (0.4859 + 2D/\kappa^{3/5})\varepsilon_F} \cong 1 - 1.215\varepsilon_F + 2.413\varepsilon$$
$$\cong 1 - 1.215\varepsilon_F. \tag{78}$$

As a conclusion, we would say that the Bose condensate occupies a thinner volume than the fermion boundary layer, and decreases in an exponential form as a function of the distance to the wall. In the two cases, at zero order in ε, the energy is equal and corresponds to the classical repulsive energy of a superficial layer adjacent to the wall of the shell. In the first order in ε, however, the energy of the fermions is higher than that of the bosons.

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References

[1] Jetzer P 1992 *Phys. Rep.* **220** 163
[2] Ruffini R and Bonazzola S 1969 *Phys. Rev.* **187** 1767
[3] Membrado M, Pacheco A F and Sañudo J 1989 *Phys. Rev.* A **39** 4207
[4] Basdevant J L, Martin A and Richard J M 1990 *Nucl. Phys.* B **343** 60
[5] Membrado M, Pacheco A F and Sañudo J 1989 *Phys. Lett.* A **135** 205
[6] Chandrasekhar S 1939 *Stellar Structure* (Chicago, IL: Dover)
[7] See, for example, Shapiro S L and Teukolsky S A 1983 *Black Holes, White Dwarfs and Neutron Stars* (New York: Wiley)
[8] See, for example, Galindo A and Pascual P 1991 *Quantum Mechanics II* (Berlin: Springer)
[9] Budker D 1998 *Am. J. Phys.* **66** 572
[10] Sañudo J and Pacheco A F 2000 *J. Phys. A: Math. Gen.* **33** 5913
[11] See, for example, Fetter A L and Walecka J D 1971 *Quantum Theory of Many-Particle Systems* (New York: McGraw-Hill)